

Avalanche Model

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Abstract

In this project we developed and completely solved a simple mathematical model that describes motion of avalanche in n dimensions. First, we considered a discrete 1-D model in which the mass of moving snow is presented in a form of “snow blocks” lined up on an inclined plane separated by a certain distance. We solved this 1-D model exactly by applying 1-D kinematic equations and using the conservation of linear momentum. With the information and data gathered from the solution of this model, we could formulate and solve a most general n -dimensional avalanche model, which is mostly applied to the realistic case of $n = 2$. As the result, we derived that the avalanche slides down with a constant acceleration $a = g \sin \theta / (1 + 2n)$, the size of the avalanche $x \sim at^2$, and the mass of the avalanche $m \sim x^n \sim t^{2n}$.

1 Introduction

An avalanche occurs when a dense layer of accumulated snow is triggered by a (natural or artificial) force or energy causing it to flow down rapidly in an inclined surface [1]. There are several factors that can contribute to the probability of an avalanche; according to the *National Snow and Ice Data Center*, wind speed and direction, steepness of slope, vegetation, terrain, the weather and temperature can increase the likelihood of such incident [2]. Although big avalanches mostly occur during and after huge snowstorms in the mountain areas, it is still considered to be one of the most dangerous catastrophes people might encounter in life, especially for mountain skiers and for people living in the mountain sides [3]. As a matter of fact, based on *Colorado Avalanche Information Centers* statistical report, from 1993 to

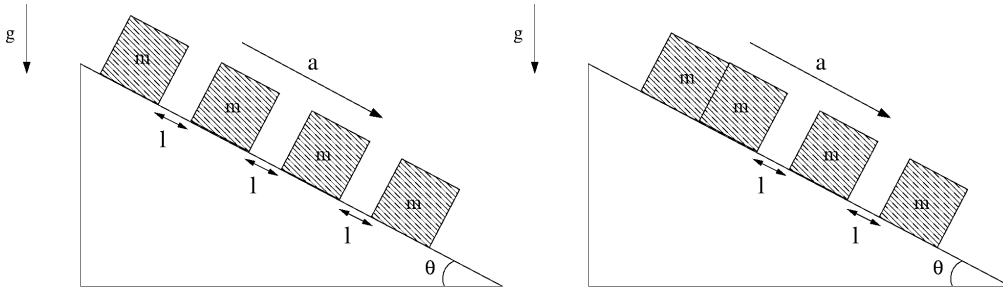


Figure 1: Motion of avalanche in 1-D model.

2016 an average of 27 fatalities in the U.S. due to avalanche has been reported [4]. In this paper, the propagation of an avalanche from the starting zone to the runout zone in 1-D, 2-D and n -D model of an inclined plane will be investigated. At the same time, give people a deeper understanding of how avalanches work, and also provide a model that can contribute to the safety of the people.

In 1-D, the avalanche model is represented as a set of blocks on an inclined plane. Each block has the same mass and distance between them. When the topmost block begins to move, it overcomes the distance to the second block; due to inelastic collision the two blocks will move as one unit of mass sliding down to the following N_{th} block, repeating the same situation making the mass increase as more blocks merge. The action of the model ends when all the blocks are one and have finished their movement with a complete stop. In the 1-D model, kinematics for constant acceleration, conservation of linear momentum, and summation was used to derive the general form of equation for initial velocity and velocity before a block hits the next block, of masses represented by N -blocks sliding down a frictionless slope. In 2-D model the avalanche model can be represented as a set of blocks on an inclined surface. The blocks are arranged in the form of a triangle. The topmost block is the angle of the triangle. The bottom of the surface is the base of the triangle. Starting from the first block, each subsequent line increases in geometrical progression. The distance between the lines of the blocks is the same. The action of the model ends when all the blocks are one and have finished their movement with a complete stop. Using the information we acquired from the first model, we introduced three postulates to solve an n -dimensional model describing the motion of an avalanche.

2 Theory Outlines

We start with a 1-D discrete model by finding the velocity of the avalanche as a function of the number of “snow blocks” that are involved into the motion. To do so, two different velocities will be introduced: V_N and U_N . V_N is the velocity of N blocks just right before they hit the next $(N + 1)$ block; U_N is the initial velocity of N blocks which they acquired after $(N - 1)$ blocks hit the N^{th} block, see Figure 1. During the time while N blocks accelerates down and travel the distance l they will increase their velocity according to the standard constant acceleration motion equation

$$V_N^2 = U_N^2 + 2al, \quad (1)$$

where l is the distance between the blocks and a is the acceleration in the system, for a frictionless motion $a = g \sin \theta$, where θ is the inclination angle.

The snow blocks have a linear momentum as they accelerate down the plane towards the following block. During the collision, the total linear momentum of the system is conserved. So we can write:

$$m_{N+1}U_{N+1} = m_N V_N, \quad (2)$$

where m_{N+1} is the mass of the $(N + 1)$ blocks, and U_{N+1} is its initial velocity; similarly, m_N is the mass of the first N blocks and V_N is its velocity just before it hits the next $(N + 1)^{th}$ block. From the Eq. (2) for a 1-D model: $m_N = m_0 N$, the initial velocity of $(N+1)$ blocks before the collision can be written as

$$U_{N+1}^2 = \frac{N^2}{(N + 1)^2} V_N^2. \quad (3)$$

To find U_N and V_N as a function of N , a certain pattern can be determined. We solve Eqs.(1) and (3) as a system of two recursive equations with the $U_1 = 0$ initial condition. The solution can be sketched as follows (for the

first four blocks):

$$\begin{aligned}
U_1^2 &= 0, \\
V_1^2 &= 2al, \\
U_2^2 &= \frac{1^2}{2^2}V_1^2 = \frac{1}{2^2}2al, \\
V_2^2 &= \frac{1}{2^2}2al + 2al = 2al\left(\frac{1}{2^2} + 1\right), \\
U_3^2 &= \frac{2^2}{3^2}V_2^2 = \frac{2^2}{3^2}\left(\frac{1}{2^2}2al + 2al\right) = 2al\left(\frac{1^2}{3^2} + \frac{2^2}{3^2}\right), \\
V_3^2 &= 2al\left(\frac{1^2}{3^2} + \frac{2^2}{3^2} + \frac{3^2}{3^2}\right), \\
U_4^2 &= 2al\left(\frac{1^2}{4^2} + \frac{2^2}{4^2} + \frac{3^2}{4^2}\right), \\
V_4^2 &= 2al\left(\frac{1^2}{4^2} + \frac{2^2}{4^2} + \frac{3^2}{4^2} + \frac{4^2}{4^2}\right),
\end{aligned} \tag{4}$$

The pattern can be easily recognized, so for V_N we can conclude:

$$V_N^2 = 2al\left(\frac{1^2}{N^2} + \frac{2^2}{N^2} + \frac{3^2}{N^2} + \dots + \frac{N^2}{N^2}\right), \tag{5}$$

or

$$V_N = \sqrt{\frac{2al}{N^2}(1^2 + 2^2 + 3^2 + \dots + N^2)} = \sqrt{\frac{2al}{N^2} \cdot \frac{N(N+1)(2N+1)}{6}}, \tag{6}$$

where we used the well known formula for the sum of the squares of the first N natural numbers, finally we got

$$V_N = V_1\sqrt{\frac{(N+1)(2N+1)}{6N}}, \tag{7}$$

$$U_N = V_1\sqrt{\frac{(N-1)(2N-1)}{6N}}, \tag{8}$$

where $V_1 = \sqrt{2al}$. One of the conclusions we can derive from the Eqs. (7) and (8) is the correlation between the avalanche's velocity and its mass, at large N :

$$V^2 \sim N \sim m, \tag{9}$$

where m is a mass of the avalanche. In 1-D model the mass the avalanche is proportional to the length of the avalanche $m \sim x$, so we can conclude

$$V^2 \sim x, \tag{10}$$

which corresponds to a constant acceleration motion.

Using the information gathered above, we can formulate three postulates that will help us to describe the motion of an avalanche in n dimensions:

- (1) The proportionality of mass to its length give us a *constant*; thus, we can write the equation:

$$m = \text{constant} \cdot x^n = c \cdot x^n, \quad (11)$$

where n is the dimension of the model. For example, in two dimensions $m \sim x^2$.

- (2) The velocity of the avalanche model can be written as the rate of change of the position with respect to time:

$$V = \frac{dx}{dt}. \quad (12)$$

- (3) The avalanche can be described using the second Newton's law as

$$\frac{dp}{dt} = mg \sin \theta, \quad (13)$$

where dp/dt is the rate at which the momentum of the avalanche changes, $mg \sin \theta$ is net force acting on the avalanche, θ is the inclination angle, and g is the free fall gravity acceleration.

Now let us solve the model we formulated above. The momentum of the avalanche p is

$$p = mV = \text{constant} \cdot x^n V. \quad (14)$$

We can rewrite the second Newton's law (13) as

$$\frac{dp}{dt} = \frac{d(mV)}{dt} = mg \sin \theta \quad (15)$$

If we substitute the mass of the avalanche as cx^n and cancel the constant c we will get

$$\frac{d(x^n V)}{dt} = x^n g \sin \theta. \quad (16)$$

This is a differential equation which can be solved by multiply the both sides of the equation by $x^n V$:

$$(x^n V) \frac{d(x^n V)}{dt} = g \sin \theta (x^{2n} V). \quad (17)$$

The left and right sides of this equation can be integrated as

$$(x^n V) \frac{d(x^n V)}{dt} = \frac{d(x^n V)^2}{2} \quad (18)$$

and

$$(x^{2n}) \frac{dx}{dt} = \frac{1}{2n+1} \frac{d}{dt} (x^{2n+1}). \quad (19)$$

Finally after integration we get

$$\frac{(x^n V)^2}{2} = \frac{g \sin \theta}{1+2n} x^{2n+1} + \text{constant}, \quad (20)$$

where *constant* is the constant of integration. At the very begin of the avalanche motion the avalanche's velocity should be zero, the same as the avalanche length, therefore we conclude that *constant* = 0.

After we cancel x^{2n} on the both sides of equation (20) we get

$$\frac{V^2}{2} = \frac{g \sin \theta}{1+2n} x. \quad (21)$$

To find the acceleration of the avalanche we can take the time derivative from the both sides of equation (21):

$$V \cdot \frac{dV}{dt} = \frac{g \sin \theta}{1+2n} \cdot \frac{dx}{dt}. \quad (22)$$

As the result we got the avalanche slides down with a constant acceleration rate $a = dV/dt$

$$a = \frac{g \sin \theta}{1+2n}. \quad (23)$$

It is interesting to note that the acceleration we got is lower than the naive $g \sin \theta$ by a factor of $(2n+1)$, this is due to the fact that while avalanche slides down its mass increases.

Finally the time dependence of the velocity, length, and mass of the avalanche is:

$$v = a \cdot t \sim t, \quad (24)$$

where velocity increases linearly with time t ;

$$x = \frac{a \cdot t^2}{2} \sim t^2, \quad (25)$$

where the size of the avalanche increases as t^2 ; and

$$m = c \cdot \left(\frac{at^2}{2}\right)^n \sim t^{2n}, \quad (26)$$

where the mass increases as t^{2n} .

3 Results

During the process of deriving the general equation for 1-D avalanche model, it was found that there's a dependence between avalanche's velocity and its mass. In other words, the mass increases as V^2 as follows from Eq. (9). Furthermore, we proved that the mass of avalanche is proportional to the size of the avalanche. For the fact that as the blocks descend down the plane due to gravity, it will gain momentum and will accumulate mass as it merges with the blocks along its path. This accumulation of mass will increase the size of the avalanche. We use this idea to formulate and solve a n -dimensional avalanche model, where mass increases as the avalanche size in power n , where n is the dimension of the model. By applying this idea, along with velocity as a rate of change in position over time, and the impulse-momentum theorem, acceleration of the avalanche was found. As the result, we derived that the avalanche slides down with a constant acceleration $a = g \sin \theta / (1 + 2n)$, the size of the avalanche $x \sim at^2$, and the mass of the avalanche $m \sim x^n \sim t^{2n}$.

What is the dimension of a realistic avalanche? At the first glance, any avalanche should be considered as a two dimensional ($n = 2$) system, in this case its acceleration will be

$$a = \frac{1}{5}g \sin \theta \quad (27)$$

and this is the main result of this work. However, it is possible to introduce a fractional dimension of the avalanche. For example, the effect of compressibility of snow may change the effective dimension of the avalanche as

$$m \sim \rho(x)x^{2n}, \quad (28)$$

where $\rho(x)$ is the density of the snow in the avalanche, which may depend on the avalanche size x . From the most general point of view, the snow in the

avalanche can be compressed as the avalanche moves down and increases in size as

$$\rho(x) \sim x^\epsilon, \quad (29)$$

where ϵ is a positive constant defining the compressibility of the snow. If this is the case, the avalanche acceleration will be

$$a = \frac{1}{5 + \epsilon} g \sin \theta. \quad (30)$$

Another opportunity is to introduce an actual fractional dimension of the avalanche and try to describe the avalanche as a fractal. Both of these opportunities are subjects for the future studies.

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