

Dark Matter in the Solar System

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Abstract

We study the effect of dark matter on the planetary motion in our Solar system. In our Solar system, it has been observed that the perihelion of many planets has been shifted by some angle but nobody could explain this why this is happening. In our research, we explained this through dark matter. This might be because of dark matter present in our solar system.

1 Introduction

Roughly 80 percent of the mass of the universe is made up of material that scientists cannot directly observe. The matter that cannot be directly observed is known as dark matter. But it can be observed indirectly by observing the motion of stars and galaxies. There are many experiments and phenomena that prove the existence of dark matter. The other evidence is the velocity of planets and stars in a galaxy. The velocity profile of stars in a galaxy should obey the third Kepler's law

$$v^2 r = \text{const} \text{ or } v \sim \frac{1}{\sqrt{r}}, \quad (1)$$

where r is the distance of the star from the center of the galaxy. In spite of Eq.(1) there are observations showing that the star velocity is not decreasing with the distance, but flattens reaching a constant value.

2 Distribution of Dark Matter

At the level of our Solar system , we assume that the density of dark matter is constant. We assume that due to the Sun's gravity the Solar System Dark matter is spherically distributed around the Sun. The average density of dark matter near the solar system is approximately 1 proton-mass for every 3 cubic centimeters, which is roughly $6 * 10^{-28} kg/cm^3$.

3 Gauss Theorem

According to the Gauss theorem, "The gravitational flux through any closed surface is proportional to the enclosed mass". This states that only the dark matter present inside the closed path of the earth will exert the force on it. The mass enclosed inside the closed path of the earth will contribute the shift in its perihelion. This elliptical orbit is approximately circular. So we have approximated our shape of the orbit to circular. Equation (2) represents the mass enclosed inside the spherical orbit of the earth.

$$M_{encl.} = \int \rho_{DM} dV = \rho_{DM} \int dV = \frac{4\pi}{3} r^3 \rho_{DM}, \quad (2)$$

where ρ_{DM} is the constant density of dark matter inside the orbit of a planet.

4 Force and potential energy to due to dark matter

The equation (3) represents the force on Earth due to dark matter. And due to this force, there will be a potential energy. This potential energy will be the correction in the potential energy of the earth. Due to this correction, there will be a shift in the perihelion of the earth. The equation (4) represents the potential energy due to dark matter.

$$\vec{F} = -\frac{GM_{encl.}m}{r^2} \hat{r}, \quad (3)$$

where G is universal gravitation constant and M is mass of dark matter enclosed inside the orbit of earth and m is the planetary mass.

$$F_{DM} = \frac{G4\pi r^3/3\rho_{DM}m}{r^2} = \frac{4}{3}\pi G\rho_{DM}mr. \quad (4)$$

Correction to the planet potential energy due to dark matter can be calculated as

$$U_{DM} = \int_0^r F_{DM} dr = \gamma r^2, \quad (5)$$

where $\gamma = 2\pi m G \rho_{DM}/3$.

5 Change in planetary motion due to Dark Matter

The shape of planetary orbits can be derived from the potential energy of the planet. Considering a planet moving on a plane its velocity can be split into two component: one is the radial component, which contributes to the radial kinetic energy and the angular one

$$E = \frac{1}{2}mv^2 + V(r) = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\theta}^2}{2} + V(r), \quad (6)$$

where $V(r)$ is the potential energy. The angular part of the kinetic energy can be “moved” to the effective potential energy as

$$E = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\theta}^2}{2} + V(r) = \frac{m\dot{r}^2}{2} + \frac{L^2}{2mr^2} + V(r), \quad (7)$$

where we used the angular momentum conservation as

$$L = mr^2\dot{\theta} = \text{const.}$$

Thus for only radial motion

$$V_{eff} = \frac{L^2}{2mr^2} + V(r) \quad (8)$$

plays a role of effective potential energy. Here L is angular momentum and it is conserved for any central force $U(r)$.

The Fig(1) between the effective potential energy and radius tell us the about the maximum and minimum distance of the planet from the center.

The Equation(7) represents the total energy. Equation (8) represents effective potential energy of planets. When we take dark matter into account there is will correction in effective potential energy. This correction in potential will result in the change of the perihelion of the earth.

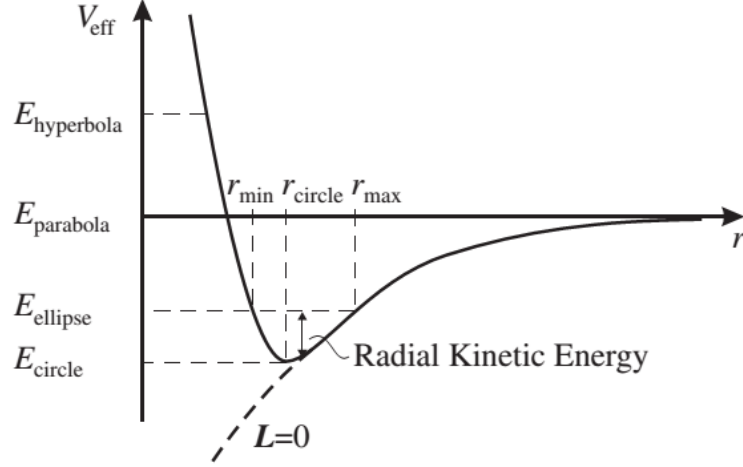


Figure 1: Effective potential energy for radial motion.

Planetary motion of the earth without dark matter In the motion of earth, the gravitational force due to sun provides the potential energy which contribute in the effective potential energy. Equation(9) represent the effective potential energy of the earth.

$$V_{eff} = \frac{L^2}{2mr^2} - \frac{\alpha}{r} \quad (9)$$

where $\alpha = GMm$ where M is mass of the sun and m is mass of the earth. The equation(10) represent the total energy of the earth.

$$E = \frac{m\dot{r}^2}{2} + \frac{L^2}{2mr^2} - \frac{\alpha}{r} \quad (10)$$

The equation(11) is about conservation of angular momentum. Due to the conservation of momentum, we are able to write our equations in term of angle which is represented in the equation(12).

$$L = mr\dot{\theta} \quad (11)$$

$$\theta = \int \frac{L}{mr^2} dt \quad (12)$$

From equation(10),by simplyfying,

$$dt = \frac{dr}{\sqrt{\frac{2}{m}(-|E| - \frac{L^2}{2mr^2} + \frac{\alpha}{r})}} \quad (13)$$

By using equation(12) and equation(13), we have

$$\theta = 2 \int_{r_{min}}^{r_{max}} \frac{L}{mr^2} \frac{dr}{\sqrt{\frac{2}{m}(-|E| - \frac{L^2}{2mr^2} + \frac{\gamma}{r})}} \quad (14)$$

where r_{min} and r_{max} are defined when $\dot{r} = 0$
by solving equation(14),

$$\theta = 2\pi$$

For details see Appendix A.

Planetary motion of the earth with dark matter When we take dark matter in account , there is gravitational force due to dark matter on earth . This gravitational force contribute in potential energy. This potential energy will be correction to effective potential energy of the earth. Equation(15) represent the effective potential energy due to dark matter. This correction will result in the change of the perihelion of the earth.

$$V_{eff} = \frac{L^2}{2mr^2} - \frac{\alpha}{r} + \gamma r^2 \quad (15)$$

where αr^2 is correction to effective potential energy due to dark matter and $\alpha = GMm$ and $\gamma = 2\pi m G \rho_{DM}/3$.

$$\theta_{DM} = 2 \int_{r_{min}}^{r_{max}} \frac{L}{mr^2} \frac{dr}{\sqrt{\frac{2}{m}(-|E| - \frac{L^2}{2mr^2} + \frac{\alpha}{r} - \gamma r^2)}} \quad (16)$$

Equation (17) represent the change in angle due to dark matter.

$$d\theta = \theta_{(\gamma \neq 0)} - \theta_{(\gamma = 0)} \quad (17)$$

by simplifying,

$$\theta = 2m\gamma \frac{d}{dL} \left[\frac{1}{L} \int \frac{dt}{t^4(\beta^2 - (t - t_0)^2)^{\frac{1}{2}}} \right] \quad (18)$$

where $t_0 = \frac{m\alpha}{L^2}$, $\beta^2 = \frac{m^2\alpha^2}{L^4} - \frac{2m|E|}{L^2}$
by solving equation(18),

$$d\theta \approx \frac{M_{dm}}{2M} \quad (19)$$

where , M_{dm} is the mass of dark matter enclosed in the orbit of planet and M is the mass the Sun.

For detail see Appendix B.

6 Result

To calculate the change in the perihelion of the earth. Since the density of dark matter in our solar system is 6×10^{-28} .The approximated density of dark matter inside the earth orbit is 6×10^{-34} .

$$M_{dm} = \rho_{dm}V$$

where ρ_{dm} is the density of density of dark matter inside the earth orbit and V is the volume of the orbit which is approximately spherical.

$$d\theta \approx \frac{M_{dm}}{2M}$$

$$d\theta = 0.42 \times 10^{-12} \text{ arcsec in 100 years}$$

On the other hand , we know the experimental values of the change in perihelion of some planets.From that we estimated the value of dark matter inside there orbits.(For reference refer article 1)

For Mercury,

$$d\theta = 1.74 \times 10^{-10} \text{ rad}$$

$$\rho_{dm} = 0.0086 \times 10^{-10} \frac{\text{kg}}{\text{m}^3}$$

For Earth,

$$d\theta = 9.70 \times 10^{-12} \text{ rad}$$

$$\rho_{dm} = 3.17 \times 10^{-15} \frac{\text{kg}}{\text{m}^3}$$

7 Appendix A

$$E = \frac{m\dot{r}^2}{2} + \frac{L^2}{2mr^2} - \frac{\alpha}{r}$$

where $\alpha = GMm$ where M is mass of the sun and m is mass of the earth.

$$\dot{r} = \pm \sqrt{\frac{2}{m} \left(E - \frac{L^2}{2mr^2} + \frac{\alpha}{r} \right)}$$

$$\dot{r} > 0, E < 0$$

$$\frac{dr}{dt} = \sqrt{\frac{2}{m} \left(-|E| - \frac{L^2}{2mr^2} + \frac{\alpha}{r} \right)}$$

$$dt = \frac{dr}{\sqrt{\frac{2}{m} \left(-|E| - \frac{L^2}{2mr^2} + \frac{\alpha}{r} \right)}}$$

$$\dot{\theta} = \frac{L}{mr^2}$$

$$\theta = \int \frac{L}{mr^2} dt$$

$$\theta = 2 \int_{r_{min}}^{r_{max}} \frac{L}{mr^2} \frac{dr}{\sqrt{\frac{2}{m} \left(-|E| - \frac{L^2}{2mr^2} + \frac{\alpha}{r} \right)}}$$

where r_{min} and r_{max} are defined when $\dot{r} = 0$

$$u = \frac{1}{r}$$

$$du = -\frac{1}{r^2} dr$$

$$\theta = -\frac{2L}{\sqrt{2m}} \int_{r_{min}}^{r_{max}} \frac{du}{\sqrt{\frac{L^2}{2m} \left(\sqrt{-(u^2 - \frac{2m\alpha u}{L^2} + \frac{2m|E|}{L^2})} \right)}}$$

$$\theta = -2 \int_{r_{min}}^{r_{max}} \frac{du}{\left(\sqrt{-(u^2 - \frac{2m\alpha u}{L^2} + \frac{2m|E|}{L^2})} \right)}$$

$$\theta = -2 \int_{r_{min}}^{r_{max}} \frac{du}{\sqrt{(\sqrt{\frac{m^2\alpha^2 - 2m|E|}{L^2}})^2 - (u - \frac{m\alpha}{L^2})^2}}$$

let

$$\beta = \sqrt{\frac{m^2\alpha^2 - 2m|E|}{L^2}}$$

$$v = u - \frac{m\alpha}{L^2}$$

$$\theta = 2 \int_{r_{max}}^{r_{min}} \frac{du}{\sqrt{\beta^2 - v^2}}$$

since

$$\dot{r} = \beta^2 - v^2$$

$$r_{max} = \beta$$

$$r_{min} = -\beta$$

by solving

$$\theta = 2\pi$$

8 Appendix B

$$E = \frac{m\dot{r}^2}{2} + \frac{L^2}{2mr^2} - \frac{\alpha}{r} + \gamma r^2$$

$\alpha = GMm$ and $\gamma = 2\pi mG\rho_{DM}/3$.

$$\dot{r} = \pm \sqrt{\frac{2}{m}(E - U(r))}$$

$$\dot{\theta} = \frac{L}{mr^2}$$

where L is angular momentum

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \frac{\pm \sqrt{\frac{2}{m}(E - U(r))mr^2}}{L^2}$$

$$\theta = 2 \int_{r_{min}}^{r_{max}} \frac{L}{mr^2} \frac{dr}{\sqrt{\frac{2}{m}(E - U(r))}}$$

where r_{min} and r_{max} defined by $\dot{r} = 0$

$$\begin{aligned} \theta &= \frac{2L}{m} \int_{r_{min}}^{r_{max}} \frac{\frac{dr}{r^2}}{\sqrt{\frac{2}{m}(-|E| - \frac{L^2}{2mr^2} + \frac{\alpha}{r} - \gamma r^2)}} \\ \theta &= -2m \int_{r_{min}}^{r_{max}} \frac{d}{dL} \sqrt{\frac{2}{m}(-|E| - \frac{L^2}{2mr^2} + \frac{\alpha}{r} - \gamma r^2)} dr \\ \theta &= -2m \frac{d}{dL} \int_{r_{min}}^{r_{max}} \sqrt{\frac{2}{m}(E - U(r))} dr \\ d\theta &= \theta_{(\gamma \neq 0)} - \theta_{(\gamma = 0)} \end{aligned}$$

$$\theta = -2m \frac{d}{dL} \left[\int \sqrt{\frac{2}{m}(E - U(r) - du)} dr - \int \sqrt{\frac{2}{m}(E - U(r))} \right]$$

by taylor expansion

$$\int \sqrt{\frac{2}{m}(E - U(r) - du)} = \int \sqrt{\frac{2}{m}(E - U(r))} - \frac{du}{\sqrt{2m}} \frac{1}{\sqrt{E - U(r)}}$$

$$d\theta = \sqrt{2m} \frac{d}{dL} \int_{r_{min}}^{r_{max}} \frac{du}{\sqrt{E - U(r)}} dr$$

where $du = \gamma r^2$, $U(r) = \frac{L^2}{2mr^2} - \frac{\alpha}{r}$

$$d\theta = \sqrt{2m} \frac{d}{dL} \int \frac{\gamma r^2 dr}{\sqrt{-|E| - \frac{L^2}{2mr^2} + \frac{\alpha}{r}}}$$

let $t = \frac{1}{r}$, $dt = -\frac{dr}{r^2}$, $dr = -\frac{dt}{t^2}$

$$\theta = \sqrt{2m} \frac{d}{dL} \int_{r_{min}}^{r_{max}} \frac{\gamma}{t^4 \sqrt{-|E| - \frac{L^2 t^2}{2m} + \alpha t}} dt$$

where $t_0 = \frac{m\alpha}{L^2}$, $\beta^2 = \frac{m^2\alpha^2}{L^4} - \frac{2m|E|}{L^2}$

$$\theta = 2m\gamma \frac{d}{dL} \left[\frac{1}{L} \int \frac{dt}{t^4(\beta^2 - (t - t_0)^2)^{\frac{1}{2}}} \right]$$

since $\dot{r} = \beta^2 - (t - t_0)^2$,

$$r_{min} = t_0 - \beta$$

$$r_{max} = t_0 + \beta$$

$$I_{\frac{1}{2}} = \int_{t_0-\beta}^{t_0+\beta} \frac{dt}{t^4(\beta^2 - (t - t_0)^2)^{\frac{1}{2}}}$$

put $t - t_0 = \beta \sin\theta$

when

$$t = t_0 - \beta$$

$$\sin\theta = -\frac{\pi}{2}$$

$$t = t_0 + \beta$$

$$\sin\theta = \frac{\pi}{2}$$

(20)

$$I_{\frac{1}{2}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{(t_0 + \beta \sin\theta)^4}$$

$$I_{\frac{1}{2}} = -\frac{1}{6\beta^4} \frac{d^3}{dt_0^3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\left(\frac{t_0}{\beta} + \sin\theta\right)}$$

let

$$x(a) = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{d\theta}{(a + \sin\theta)}$$

where $a = \frac{t_0}{\beta}$

put

$$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$$

$$x(a) = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\sec^2 \frac{\theta}{2} d\theta}{a + a \tan^2 \frac{\theta}{2} + 2 \tan \frac{\theta}{2}}$$

put

$$\begin{aligned} y &= \tan \frac{\theta}{2} \\ 2dy &= \sec^2 \frac{\theta}{2} d\theta \end{aligned} \tag{21}$$

when

$$\begin{aligned} \theta &= \frac{\pi}{2} \\ y &= \tan\left(\frac{\pi}{4}\right) = 1 \\ \theta &= -\frac{\pi}{2} \\ y &= -\tan\left(\frac{\pi}{4}\right) = -1 \end{aligned} \tag{22}$$

$$x(a) = \int_1^{-1} \frac{2dy}{a + at^2 + 2t}$$

by solving,

$$x(a) = \frac{2}{\sqrt{a^2 - 1}} \left[\arctan\left(\frac{-a + 1}{\sqrt{a^2 - 1}}\right) - \arctan\left(\frac{a + 1}{\sqrt{a^2 - 1}}\right) \right]$$

by simplyfying,

$$\begin{aligned} x(a) &= \frac{\pi}{\sqrt{a^2 - 1}} \\ I_{\frac{1}{2}} &= -\frac{1}{6\beta^4} \frac{d^3}{da^3} (x(a)) \end{aligned}$$

by simplying

$$I_{\frac{1}{2}} = -\frac{5}{3} \frac{(m\alpha)^3 L}{(2m|E|)^{\frac{7}{2}}} + \frac{m\alpha L^3}{2m|E|^{\frac{5}{2}}}$$

$$d\theta = 2m\gamma \frac{d}{dL} \left(\frac{1}{L} I_{\frac{1}{2}} \right)$$

$$d\theta = \frac{4m^2\gamma\alpha L}{(2m|E|)^{\frac{5}{2}}}$$

by using parameters a and b , we get

$$a = \frac{\alpha}{2|E|}$$

$$b = \frac{L}{\sqrt{2m|E|}}$$

$$d\theta = \frac{4\gamma ba^2}{\alpha}$$

by substituting

$$\alpha = GMm$$

$$\gamma = \frac{2}{3}\pi Gm\rho_{dm}$$

$$d\theta = \frac{2\pi\rho_{dm}ba^2}{3}$$

$$d\theta \approx \frac{M_{dm}}{2M}$$

9 Reference

Article 1. E.V.Pitjeva Relativistic Effects and Solar Oblateness from Radar Observations of Planets and Spacecraft