

Orbital Decay of a Massive Black Hole

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Abstract

In this research, we derived starting with the first principles, the dynamical friction force on a star moving through a star field/galaxy. We used this derivation to calculate the orbital decay of a heavy Black Hole (BH) or a massive star. We found that the typical time for a BH to spiral into the galactic center, the decay time, is comparable to the age of the universe. Our estimation showed that this decay time scales with the mass of the BH as

$$t \sim 10^{18} M_{\odot}/M \text{ years,}$$

where M is the mass of the BH and $M_{\odot} = 2 \times 10^{30}$ kg is the solar mass.

1 Introduction

Dynamical friction is a force that slows down objects moving in space. It reduces the kinetic energy of interacting bodies as a result of gravitational attraction. This reduction in kinetic energy can be observed from small scattering processes, such as a comet traveling past a star, all the way up to galactic mergers. We derived the formula for dynamical friction in terms of the loss of velocity of a star passing through a star field. Afterwards, we determined the density profile of a galaxy (based on the rotational curves), and then using this density profile, estimated the orbital decay time of a heavy BH around a supermassive galactic center (a supermassive black hole or SMBH).

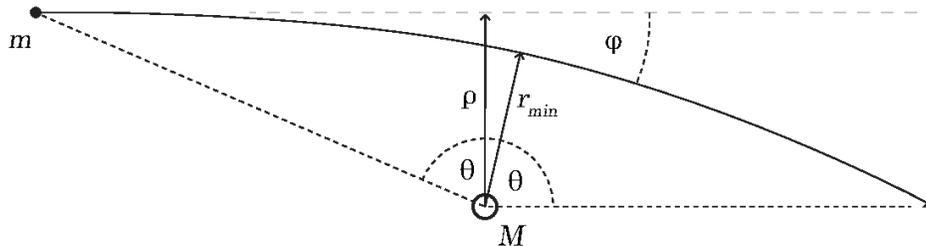


Figure 1: Scattering of mass m by a heavy target M .

2 Theory Outline

2.1 The Kepler Problem

We begin by considering the Kepler Problem; the scattering process of two point-like masses due to mutual gravitational interaction. This process can be described by the change in relative velocity of the two masses m and M (or just the change in velocity of m if $M \gg m$). If we exclude external influences, the center-of-mass velocity of the system is unchanged.

$$\begin{aligned}\Delta v_{\parallel} &= -v + v \cos \phi, \\ \Delta v_{\perp} &= v \sin \phi.\end{aligned}\tag{1}$$

Here $\vec{v} = \vec{v}_1 - \vec{v}_2$ is relative velocity, $\Delta \vec{v}$ is the change in relative velocity, and ϕ is the angle of deflection, see Fig. 1. Assuming a uniform distribution of stars in the field, changes in the perpendicular velocity component cancel out, see Fig. 2. From now on only the parallel velocity component is of interest. The scattering angle ϕ relates to the direction of closest approach between the stars, with the corresponding angle θ_0 shown in Fig. 1. Now, with the help of

$$\phi = 2\theta_0 - \pi,\tag{2}$$

we are able to define the change in velocity in terms of θ_0 as

$$\Delta v_{\parallel} = -2v \cos^2 \theta_0.\tag{3}$$

The solution of the Kepler Problem is well known—it relates the distance between two colliding bodies r with the polar angle θ as

$$r = \frac{p}{1 + e \cos \theta}.\tag{4}$$

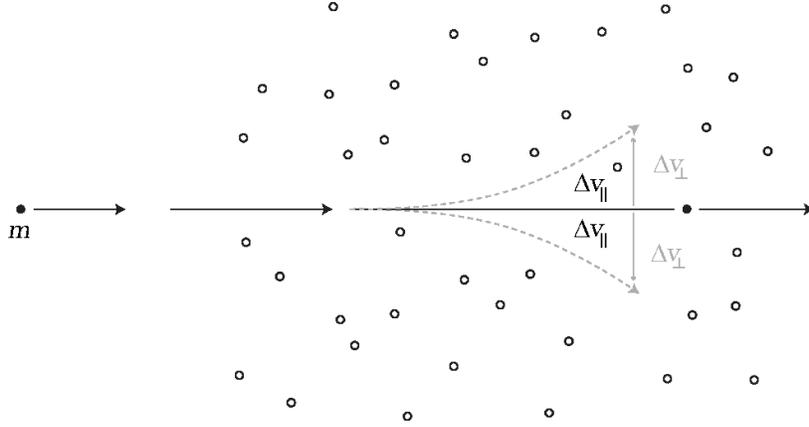


Figure 2: When moving through a uniform field, the parallel components of velocity are summed, and the perpendicular components are canceled.

Here p is the conic parameter and e is the eccentricity. Using the solution of the Kepler Problem (4) we can define angle θ_0 and the corresponding change in velocity as

$$\cos \theta_0 = -\frac{1}{e}, \text{ when } r \rightarrow \infty \quad (5)$$

and

$$\Delta v_{\parallel} = -2v \frac{1}{e^2} = -2v \frac{1}{1 + \frac{v^2 L^2}{k^2}}. \quad (6)$$

For eccentricity we used the known expression

$$e = \sqrt{1 + \frac{v^2 L^2}{k^2}}, \quad (7)$$

where $k = Gm_1 m_2$ is the strength of the gravitational interaction, $L = \mu \rho v$ is the relative angular momentum (with the impact parameter labeled as ρ , see Fig. 1), and $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass of the system.

The solution of the Kepler Problem allowed us to derive Eq.(6) without serious effort. From here, we could find the individual velocities of the scattering stars. This can be done using the following transformation (Fig. 3 may help in visualizing it)

$$\vec{v}_1 = \vec{V}_{com} + \frac{m_2}{m_1 + m_2} \vec{v}, \quad (8)$$

$$\vec{v}_2 = \vec{V}_{com} - \frac{m_1}{m_1 + m_2} \vec{v}, \quad (9)$$

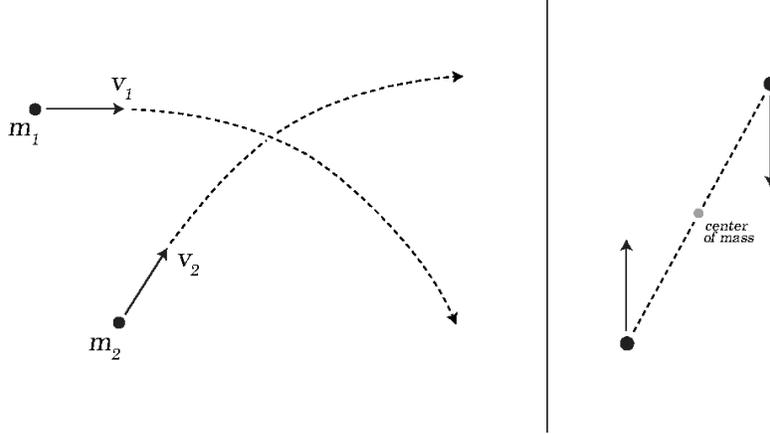


Figure 3: Center-of-mass and relative velocities.

where \vec{v}_1 is the velocity of m_1 , \vec{v}_2 is the velocity of m_2 , the center-of-mass velocity \vec{V}_{com} does not change during the scattering, and \vec{v} is the relative velocity.

Let's assign m_1 to the passing star and m_2 to the target. The change in \vec{v}_1 can be related to the change in relative velocity \vec{v} . Using Eqs.(6) and (8) we get

$$\Delta v_{1\parallel} = -2v \left(\frac{m_2}{m_1 + m_2} \right) \left(\frac{1}{1 + \frac{\rho^2 v^4}{G^2 (m_1 + m_2)^2}} \right), \quad (10)$$

where v is the relative velocity, and ρ is the impact parameter.

2.2 Dynamical Friction Force

We can integrate over the stars in the field/galaxy assuming a distribution of masses m_2 (targets) located at different positions \vec{r} , moving with different velocities \vec{v}_2 . The number of targets can be expressed through the field concentration and corresponding elementary volumes

$$dN = n(\vec{r}, \vec{v}_2) d^3 r d^3 v_2, \quad (11)$$

where $n(\vec{r}, \vec{v}_2)$ is the generalized concentration of stars of mass m_2 in the field/galaxy at point \vec{r} moving with velocity \vec{v}_2 . Note that \vec{r} is the position of m_2 relative to the passing star m_1 .

The deceleration of the passing star due to interactions with dN targets can be calculated as

$$\frac{\Delta V_{\parallel}}{\Delta t} = \frac{\Delta v_{1\parallel}}{\Delta t} dN = n(\vec{r}, \vec{v}_2) \Delta v_{1\parallel} 2\pi\rho d\rho \frac{\Delta z}{\Delta t} d^3v_2, \quad (12)$$

where we assumed cylindrical symmetry and used the cylindrical coordinates for the volume: $d^3r = 2\pi\rho d\rho dz$. With some approximation, $\Delta z/\Delta t$ can be replaced with the relative velocity parallel component v_{\parallel} .

The acceleration (12) can be integrated over the impact parameter ρ , which can be done analytically if we assume that the concentration n is more-or-less constant near the position of the passing star: $n(\vec{r}, \vec{v}_2) \approx n(0, \vec{v}_2) \equiv n_0(\vec{v}_2)$, but goes to zero for large impact parameters $\rho > \rho_{max}$. This could be, for example, the size of the galaxy. The integration over ρ gives us

$$\frac{\Delta V_{\parallel}}{\Delta t} = -2vv_{\parallel}n_0(\vec{v}_2)d^3v_2 \left(\frac{m_2}{m_1 + m_2} \right) \int \frac{2\pi\rho d\rho}{1 + \frac{\rho^2 v^4}{G^2(m_1+m_2)^2}}, \quad (13)$$

or, finally, with the logarithmic accuracy we get

$$\frac{\Delta V_{\parallel}}{\Delta t} = -4\pi G^2(m_1 + m_2)m_2 \ln \lambda \frac{v_{\parallel}}{v^3} n_0(\vec{v}_2) d^3v_2, \quad (14)$$

where $v_{\parallel} = \Delta z/\Delta t$ is the relative velocity in the direction of motion of the passing star (z is the relative coordinate), the cut-off parameter Λ is defined by the maximum impact parameter ρ_{max} after which the star field disappears: $\Lambda \sim \rho_{max}v_c^2/Gm$, and v_c is the typical speed.

Now, if we multiply acceleration (14) by the mass of the passing star m_1 we get the dynamical friction force on m_1 :

$$\vec{F}_{df} = -4\pi G^2(m_1 + m_2)m_1m_2 \ln \lambda \int n_0(\vec{v}_2) \frac{\vec{v}_1 - \vec{v}_2}{|\vec{v}_1 - \vec{v}_2|^3} d^3v_2, \quad (15)$$

where we have substituted the relative velocity with: $\vec{v} = \vec{v}_1 - \vec{v}_2$.

For further calculation we need to assume a certain distribution of stars in the field over velocity, $n_0(\vec{v}_2)$. For example, one could pick a Maxwell distribution. In our research, it would be sufficient to estimate the dynamical friction force (15) for a very heavy passing star: $m_1 \gg m_2$. For an estimation, the concentration n_0 can be replaced with the density of the target stars

$$\rho \sim m_2 \int n_0 d^3v_2, \quad (16)$$

where ρ is the star density at the location of the passing star (so $\rho = \rho(r)$ is a function of the passing star position). This gives us a rough estimation of the force;

$$F_{df} \sim 4\pi G^2 m_1^2 \rho(r) \frac{\ln \Lambda}{v_c^2}, \quad (17)$$

where G is the universal gravitational constant, m_1 is the mass of the passing star, ρ is the density of the field, v_c is the typical speed, and Λ is the cut-off parameter. For a more accurate integration of (15) the Gauss theorem can be applied. We can assume spherical symmetry for the velocity distribution $n_0(\vec{v}_2)$. In this case, the velocities in the star field exceeding that of the spiraling body do not contribute to the dynamical friction force, something of particular interest.

2.3 Orbital Decay of a Heavy BH

We can determine the density profile formula, $\rho = \rho(r)$ needed to calculate the dynamical friction force (17). In the case of spherical symmetry, the centripetal acceleration of a star moving around a galactic center is defined by the total mass enclosed by the star's orbit (another sample application of the Gauss Theorem!),

$$\frac{v^2}{r} = G \frac{\int^r \rho(x) 4\pi x^2 dx}{r^2}. \quad (18)$$

Taking the orbital speed v as constant (due to the flattening of the rotational curves) we get

$$\frac{1}{r} \int^r \rho(x) x^2 dx = constant. \quad (19)$$

It should be noted that our estimate will not take into account that rotational curves are not flat all the way to the center, but still it gives a sufficient enough estimation for the galactic density. We can see that the corresponding density profile decreases with distance as $\sim 1/r^2$, stated more accurately,

$$\rho(r) = \frac{v_c^2}{4\pi G r^2}. \quad (20)$$

We can now use Eq.(20) and the dynamical friction force (17) for a heavy passing star of mass M (in our case a heavy BH) moving through a field of stars. The dynamical friction force can be approximated as

$$F_{df} \sim 4\pi G^2 M^2 \rho(r) \frac{\ln \Lambda}{v_c^2}. \quad (21)$$

Substituting Eq.(20) into Eq.(21) we can get the following final expression for the dynamical friction force

$$F_{df} \sim \ln \Lambda \frac{GM^2}{r^2}. \quad (22)$$

The orbital decay time of a heavy BH can be described in terms of orbital radius, orbital speed, the mass of the BH, and the cut-off parameter Λ discussed above. We can estimate the decay time as the ratio of the star's angular momentum, L , to the torque, τ , exerted on the star by the dynamical friction force:

$$t \sim \frac{L}{\tau} = \frac{rMv_c}{rF_{df}} = \frac{Mv_c}{F_{df}}. \quad (23)$$

Finally, the decay time simplifies to

$$t \sim \frac{v_c r^2}{GM \ln \Lambda}. \quad (24)$$

For an estimation, we chose a BH equivalent to 10^8 solar masses ($M = 10^8 M_\odot = 2 \times 10^{38} kg$), with an orbital speed $v_c = 200 km/s = 2 \times 10^5 m/s$, an orbital radius of $r \sim 10 kpc = 3.1 \times 10^{20} m$, and a cut-off parameter of $\ln \Lambda \sim 7$. The universal gravitational constant $G = 6.67 \times 10^{-11} Nm^2/kg^2$. Using these values we get the following estimation for the decay time

$$t \sim \frac{2 \times 10^5 \cdot (3.1 \times 10^{20})^2}{6.67 \times 10^{-11} \cdot 2 \times 10^{38} \cdot 7} \approx 2 \times 10^{17} s = 6.5 \times 10^9 yr = 6.5 Gyr. \quad (25)$$

At around 7 billion years, our estimate is comparable to the life-time of the universe. The corresponding mass dependence may look like

$$t \sim 10^{18} \frac{M_\odot}{M} \text{ years}. \quad (26)$$

3 Conclusion

In this work we derived from the initial principles, the dynamical friction formula that describes an effective “friction” force acting on a star moving through a uniform star field. To derive the dynamical friction force we used the exact solution of the Kepler Problem, which allowed us to describe the scattering process of two arbitrary stars. To model the field of stars we introduced a reasonable distribution of stars in the field/galaxy $n(\vec{r}, \vec{v}_2)$.

Analyzing dynamical friction provided a means to determine the lifetime of a star's orbit around a galactic center. The lifetime of a heavy black hole orbiting a galactic center was calculated by examining the amount of frictional force applied by the star field. We found that the typical decay time of a heavy BH ($M = 10^8 M_\odot$) due to the dynamical friction force is about

$$t \sim 7 \text{ billion years}, \quad (27)$$

which is comparable to the age of our universe. A long decay time like this could suggest that heavy BHs do not spiral into the center, or do so over very long periods. Lighter stars and BHs will have an even longer decay time. Other exciting applications of dynamical friction involve simulating merging galaxies (such as Andromeda and the Milky Way), and examining how it plays a role in dark matter interactions.

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